Math 214 - Quiz 1

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Note: The set $\mathbb{R}$ of real numbers is always implicitly endowed with its Euclidean topology, unless specified otherwise.

Exercise 1. Endow $\mathbb{R}^2$ with the product topology and its subsets with the subspace topology. Let A be the open interval lying on the x-axis and joining the point $(-1,0)$ and $(1,0)$. True or False? Justify.

1. A is open in $\mathbb{R}^2$. No
2. A is open in the x-axis of $\mathbb{R}^2$. Yes $A = \underset{(\text{open})}{S^2} \cap (x\text{-axis})$
3. A is closed in the open unit disk of $\mathbb{R}^2$.

Exercise 2. Let X be a topological space, A a subset of X and U an open subset of X.

1. Show that the closure of $A \cap U$ in U (with its subspace topology) is equal to $\overline{A} \cap U$, where $\overline{A}$ is the closure of A in X. Deduce that if A is dense in X, then $A \cap U$ is dense in U.
2. Do the results of the previous question remain true if U fails to be open in X?

Exercise 3. Recall that if X is a topological space and $x \in X$, then

- a *neighborhood* of x is any subset of X containing an open subset containing x
- a *closed neighborhood* F of X is any neighborhood of x which is itself closed in X, i.e. F is closed in X and F contains an open neighborhood of x.

For example, for $X = \mathbb{R}$ and $x = 0$, then neither $\{0\}$ nor $\mathbb{Z}$ are closed neighborhood of 0 (although they are both closed in $\mathbb{R}$) while $[-1,1] \cup \{2\}$ is a closed neighborhood of 0 (because it is itself closed in $\mathbb{R}$ and it contains $U = (-1,1)$ which is an open neighborhood of 0).

1. Let X be a topological. Show that the following statements are equivalent:
   (a) X is Hausdorff
   (b) For every $x \in X$, the intersection of all closed neighborhoods of x in X is $\{x\}$.
2. Give an example of a T_1-space for which statement (b) above fails. Justify quickly your claims.

Exercise 4. Let X be a topological space, I a set and $(A_\alpha)_{\alpha \in I}$ a collection of subsets of X.

1. (a) Check that $\bigcup_{\alpha \in I} \overline{A_\alpha} \subseteq \overline{\bigcup_{\alpha \in I} A_\alpha}$.
   (b) Show that the other inclusion holds if I is a finite set. Indicate where you are using the finiteness assumption on I in your proof.
   (c) Show, using a counter-example, that the other inclusion of part (a) may however fail if I is an infinite set.
   (d) Criticize the following wrong "proof" that $\overline{\bigcup_{\alpha \in I} A_\alpha} \subseteq \bigcup_{\alpha \in I} \overline{A_\alpha}$: "*Let $x \in \overline{\bigcup_{\alpha \in I} A_\alpha}$. Then every neighborhood U of x intersects $\bigcup_{\alpha \in I} A_\alpha$. Thus U must intersect some A_α, so that x must belong to the closure of some A_α. Therefore $x \in \bigcup_{\alpha \in I} \overline{A_\alpha}$* ".

2. **Definition (Locally finite collection):**
Let X be a topological space. A collection $\mathcal{A} = (A_\alpha)_{\alpha \in I}$ of subsets of X is said to be *locally finite* in X if every point in X has an open neighborhood that intersects only finitely many elements of $\mathcal{A}$ (i.e. for every $x \in X$, there exists U open in X containing x and there exists $r \in \mathbb{N}$, $\alpha_1, \cdots, \alpha_r \in I$ such that $U \cap A_{\alpha_i} = \emptyset$ for all $i \in I \setminus \{\alpha_1, \cdots, \alpha_r\}$).

   (a) Examples:

      i. Check that $\mathcal{A} := \{[n, n+1]; n \in \mathbb{Z}\}$ is locally finite in $\mathbb{R}$.

      ii. Is $\mathcal{A} := \{[-n, n]; n \in \mathbb{N}\}$ locally finite in $\mathbb{R}$?

      iii. Check that $\mathcal{A} := \{(0, 1/n); n \in \mathbb{N}^*\}$ is locally finite in $(0,1)$ (endowed with the subspace topology) but not in $\mathbb{R}$.

   (b) i. Show that if $\mathcal{A}$ is locally finite in X then the collection $\mathcal{A}' = \{\overline{A}; A \in \mathcal{A}\}$ of the closures in X of elements of $\mathcal{A}$ is also locally finite.

      ii. In this question $X = \mathbb{R}$ endowed with the cofinite topology $\mathcal{T}_f$. Let $\mathcal{A} = \mathcal{T}_f$, i.e. $\mathcal{A}$ is the collection of all open subsets in $(\mathbb{R}, \mathcal{T}_f)$. Show, using $(\mathbb{R}, \mathcal{T}_f)$ and $\mathcal{A}$, that the converse of the result of the previous question is not true in general.

   (c) Let $\mathcal{A} = (A_\alpha)_{\alpha \in I}$ be a locally finite collection of subsets of X. Show that

$$\bigcup_{\alpha \in I} \overline{A_\alpha} = \overline{\bigcup_{\alpha \in I} A_\alpha}.$$

   Deduce that the union of closed subsets of X that form a locally finite collection remains closed in X.

Exercise 5. Let X be the set of all functions from $[0,1]$ to $\mathbb{R}$. For every $f \in X$, for every non zero integer N, for every choice of N points $x_1, \cdots, x_N$ in $[0,1]$ and for positive real number $\epsilon > 0$, we define the following subset of X:

$$B_{f, x_1, \cdots, x_N, \epsilon} := \{g \in X; \forall i = 1, \cdots, N, |f(x_i) - g(x_i)| < \epsilon\}.$$

For example if $f \in X$, $x = 0$ and $\epsilon = 0.1$, then $B_{f,x,\epsilon}$ is nothing than the set of all functions $g : [0,1] \longrightarrow \mathbb{R}$ such that $|g(0) - f(0)| < 0.1$.

Let $\mathcal{B} \subseteq \mathcal{P}(X)$ be the collection of subsets of X that consists of all the $B_{f, x_1, \cdots, x_N, \epsilon}$ with f varying in X, N varying in $\mathbb{N}^*$, $x_1, \cdots, x_N$ varying in $[0,1]$ and ϵ varying in $(0, +\infty)$.

1. Show that $\mathcal{B}$ is a basis for some topology $\mathcal{T}$ on X.
   Advice: Begin by understanding what does it mean for two elements $B_1, B_2 \in \mathcal{B}$ to have an empty intersection.

2. Is $A := \{f \in X; \forall x \in [0,1], f(x) < 1\}$ open in $(X, \mathcal{T})$?

3. Show that $(X, \mathcal{T})$ is Hausdorff.

4. Let $D \subseteq X$ be the set all functions $g : [0,1] \longrightarrow \mathbb{R}$ such that $\{x \in [0,1]; g(x) \neq 0\}$ is finite. Show that D is dense in $(X, \mathcal{T})$.

5. Show that a sequence $(f_n)_{n \in \mathbb{N}}$ in X converges to some f in $(X, \mathcal{T})$, if and only if, for every $x \in [0,1]$ the numerical sequence $(f_n(x))_{n \in \mathbb{N}}$ converges to $f(x)$ in $\mathbb{R}$.

6. Let f be any function $[0,1] \longrightarrow \mathbb{R}$ such that f is non zero on an uncountable subset of $[0,1]$ (like for instance the function $f(x) = \exp(x)$, $x \in [0,1]$).
   Show that there is no sequence in D converging to f in $(X, \mathcal{T})$. Is $(X, \mathcal{T})$ first-countable?

Exercise 6. (Bonus) Let $(X, \mathcal{T})$ be a second countable topological space.

1. Show that $\mathcal{T}$ has at most the cardinality of the continuum, i.e. that there exists an injection from $\mathcal{T}$ into $\mathbb{R}$.
   Hint: It is enough to find an injection from $\mathcal{T}$ to the set $\{0,1\}^{\mathbb{N}}$ of all functions from $\mathbb{N}$ to $\{0,1\}$. Indeed, $\{0,1\}^{\mathbb{N}}$ is in bijection with $\mathbb{R}$ (no need to prove this fact).

2. Assume that X is moreover a T_1-space. Use the result of the previous question, to show that X itself has at most the cardinality of the continuum.